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A Galactic Model for Gravitational Radiation

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Abstract

Using the linearized theory of general relativity the gravitationally radiated energy emitted from a galactic model consisting of N gravitational radiators is calculated. The results are presented in terms of the lowest order contributing multipole moments (quadrupole), the orientations of radiators about a common reference frame, the distances between pairs of radiators and the frequency of each radiator. From this model it is hoped that a reasonable lower bound to the gravitational energy flux from the galactic core can be computed.

1. *Introduction*

In recent times much interest has been shown in the results of Professor Weber's experiments to detect gravitational radiation (Weber, 1969). He claims that his results indicate the existence of energy carrying gravitational radiation and a probable source being the galactic core (Weber, 1971). To date there are no adequate models of a galaxy that can be used to determine a quantitative analysis of gravitational energy flux. This paper presents a galactic model that can be used for the calculation of gravitational radiation from such a source.

Einstein (1918) initiated the study of gravitational radiation when he calculated the energy flux emanating from a single isolated source within the framework of the linearized theory of general relativity (Landau & Lifshitz, 1965). He found that, to quadrupole order, the power loss is

$$\dot{E} = -\frac{G}{45c^5}\dddot{D}^{\alpha\beta}\dddot{D}^{\alpha\beta} \tag{1.1}$$

where

$$D^{\alpha\beta} = \int \rho(3\xi^\alpha\xi^\beta - \delta_\alpha{}^\beta\xi^\gamma\xi^\gamma)\,d^3\xi \tag{1.2}$$

is the quadrupole moment of the source.

Recently, Cooperstock & Booth (1969) presented a paper in which was calculated the total power loss from a binary system and found that not

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only do the individual sources radiate independently of each other but that there is also a power loss due to their mutual interaction. Unfortunately, as a consequence of the restricted orientation imposed on the relative separation of the two sources in the CB binary system, a galactic model cannot be built by adding the effects of pairs of arbitrarily orientated sources.

Here the problem is attacked anew and the restrictive orientation condition is omitted. In Section 2 the interaction power emitted by an arbitrary radiator pair is developed and the galactic model is taken as the algebraic sum of such pairs. A discussion follows in Section 3.

2. *Interaction Power*

The energy-momentum conservation laws in general relativity for a material stress-energy distribution T^{ik} are given as†

$$T^{ik}{}_{;k} = 0 \tag{2.1}$$

and can be expressed in the form of an ordinary divergence

$$[(\sqrt{-g})(T_i{}^k + t_i{}^k)]_{,k} = 0 \tag{2.2}$$

where the energy-momentum pseudotensor (Møller, 1966)

$$(\sqrt{-g})\, t_i{}^k = \frac{c^4}{16\pi G}[(\partial \mathscr{L}/\partial g^{lm}{}_{,k})\, g^{lm}{}_{,i} - \delta_i{}^k \mathscr{L}] \tag{2.3}$$

$$\mathscr{L} = (\sqrt{-g})\, g^{ik}[\Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^l_{im}\Gamma^m_{kl}] \tag{2.4}$$

For weak fields (Weber, 1961; Trautman, 1962)

$$(\sqrt{-g})\, t_i{}^k = \frac{c^4}{64\pi G}[(2\Psi_a{}^b{}_{,i}\, \Psi_b{}^{a,k} - \Psi_a{}^a{}_{,i}\, \Psi_b{}^{b,k}) + \delta_i{}^k(\tfrac{1}{2}\Psi_a{}^a{}_{,c}\, \Psi_b{}^{b,c} - \Psi_{ab,c}\, \Psi^{ab,c})] \tag{2.5}$$

where

$$\Psi_i{}^j = \frac{4G}{c^4}\int T_i{}^j(t - R/c)\frac{d^3\xi}{R} \tag{2.6}$$

are the retarded potential solutions to Einstein's linearized field equations.

Setting $i = 0$ in equation (2.2), integrating over a volume V which contains the material distribution and applying the Gauss theorem yields the total energy-loss rate

$${}_{\text{tot}}\dot{E} = -c\oint(\sqrt{-g})\, t_0{}^\alpha n_\alpha\, dS \tag{2.7}$$

† A semicolon denotes covariant differentiation.

Taking the material distribution to consist of N radiators,† then if $\overset{(i)}{\psi}{}_a{}^b$ denotes the field of the ith radiator, the total field is

$$\Psi_a{}^b = \sum_{i=1}^{N} \overset{(i)}{\psi}{}_a{}^b \tag{2.8}$$

From equations (2.5) and (2.8) it can be seen that equation (2.7) consists of two types of integral; one type containing terms quadratic in the field of a given radiator and the other containing products of fields of pairs of radiators. The sum of all integrals of the former type yields the power loss of the radiators in the absence of interaction and the latter integrals yield the interaction power between pairs of radiators.

The interaction power between the pth and the qth radiators is found to be

$$\dot{E}_{pq} = \frac{-c^5}{64\pi G} \oint \Big\{ 2\overset{(p)}{\psi}{}_a{}^b{}_{,0} \overset{(q)}{\psi}{}_b{}^{a,\alpha} + 2\overset{(q)}{\psi}{}_a{}^b{}_{,0} \overset{(p)}{\psi}{}_b{}^{a,\alpha} - \overset{(p)}{\psi}{}_a{}^a{}_{,0} \overset{(q)}{\psi}{}_b{}^{b,\alpha} - \overset{(q)}{\psi}{}_a{}^a{}_{,0} \overset{(p)}{\psi}{}_b{}^{b,\alpha} \Big\} n_\alpha \, dS \tag{2.9}$$

Since equation (2.9) can be evaluated over the infinite sphere the use of asymptotic fields is justified for arbitrary radiator pair separations L_{pq}.

The retarded potential solution of Einstein's linearized field equations for a radiator field is

$$\bar{\psi}^{ik} = \frac{4G}{c^4} \int \bar{T}^{ik}(\bar{\xi}^\alpha, t - \bar{R}/c) \frac{d^3\bar{\xi}}{\bar{R}} \tag{2.10}$$

where $\bar{\xi}^\alpha$ are spatial sources variables and $\bar{R}$ is the source-point to field-point distance.‡ The geometrical arrangement is illustrated in Fig. 1. We have

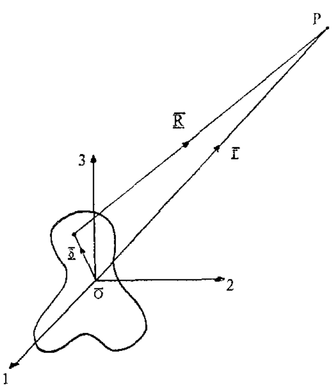

FIGURE 1

† The word 'radiator' is used throughout to denote a material stress-energy distribution.
‡ The bar over the quantities in equation (2.10) imply a specific radiator.

$$\bar{R}^2 = (\bar{x}^\alpha - \xi^\alpha)^2 = \bar{r}^2 - 2\bar{x}^\alpha \xi^\alpha + \xi^\alpha \xi^\alpha \tag{2.11}$$

where

$$\bar{r} = (\bar{x}^\alpha \bar{x}^\alpha)^{1/2}$$

is the distance from the origin (placed at the centre of mass) to the field point P.

The $\bar{T}^{ik}$ are now expanded about the retarded time $t - \bar{r}/c$

$$\bar{T}^{ik}(\xi^\alpha, t - \bar{R}/c) = \bar{T}^{ik} + (\bar{r} - \bar{R})\bar{T}^{ik}{}_{,0} + \tfrac{1}{2}(\bar{r} - \bar{R})^2 \bar{T}^{ik}{}_{,00} + \dots \tag{2.12}$$

where $\bar{T}^{ik}$, $\bar{T}^{ik}{}_{,0}$, $\bar{T}^{ik}{}_{,00} \dots$ are evaluated at $t - \bar{r}/c$. For the calculation of asymptotic fields, it can be taken that since

$$\frac{\bar{x}^\alpha}{\bar{r}} \equiv \bar{n}_\alpha \qquad \text{and} \qquad \underline{\bar{R}} = \bar{r} - \xi \tag{2.13}$$

then

$$\begin{aligned} \bar{r} - \bar{R} &\simeq \bar{n}_\alpha \xi^\alpha \\ \bar{R}^{-1} &\simeq \bar{r}^{-1} \end{aligned} \tag{2.14}$$

Equations (2.12) to (2.14) in conjunction with equation (2.10) yield

$$\bar{\psi}^{ik} = \frac{4G}{c^4 \bar{r}} \int [\bar{T}^{ik}(\xi^\alpha, t - \bar{r}/c) + \bar{n}_\alpha \xi^\alpha \bar{T}^{ik}{}_{,0}(\xi^\alpha, t - \bar{r}/c) + \dots] \, d^3\xi \tag{2.15}$$

Equation (2.15) gives a radiator field expanded about a retarded time relative to its own centre of mass. Since we have a number of radiators it is required to expand the field of each about a common retarded time;

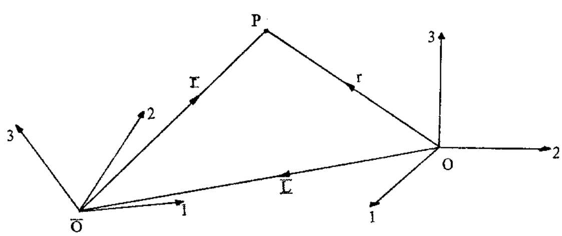

FIGURE 2

the common retarded time being relative to a chosen common origin O. From Fig. 2 it is seen that, for asymptotic fields, since

$$\bar{r} = \underline{r} - \underline{L} \tag{2.16}$$

that

$$\begin{aligned} \bar{r} &\simeq r - \underline{L}.\underline{n} \\ \bar{n}_\alpha &\simeq n_\alpha \\ \bar{r}^{-1} &\simeq r^{-1} \end{aligned} \tag{2.17}$$

Consequently equation (2.15) can be written as

$$\bar{\psi}^{ik} = \frac{4G}{c^4 r}\int [\bar{T}^{ik}(\xi^\alpha, t - r/c + \underline{L}.\underline{n}/c) + n_\alpha \xi^\alpha \bar{T}^{ik}{}_{,0}(\xi^\alpha, t - r/c + \underline{L}.\underline{n}/c) + \ldots]\, d^3\xi \tag{2.18}$$

Following Papapetrou (1962) and CB, the time-dependent field components to lowest multipole order can be simply expressed as

$$\bar{\psi}^{\alpha\beta} = \frac{4G}{c^4 r}\bar{M}^{\alpha\beta} \tag{2.19}$$

where

$$\bar{M}^{\alpha\beta} = \int \bar{T}^{\alpha\beta}\, d^3\xi = \tfrac{1}{2}\bar{d}^{\alpha\beta}{}_{,00} \qquad \text{and} \qquad \bar{d}^{\alpha\beta} = \int \bar{T}^{00}\xi^\alpha \xi^\beta\, d^3\xi \tag{2.20}$$

Also

$$\bar{\psi}^{\alpha 0} = \bar{\psi}^{\alpha\gamma} n_\gamma \qquad \text{and} \qquad \bar{\psi}^{00} = \bar{\psi}^{\alpha\beta} n_\beta n_\alpha \tag{2.21}$$

To evaluate the interaction power, equation (2.9), the integrand is required to order r^{-2}, hence it is sufficient to set

$$\bar{\psi}^{ij}{}_{,\alpha} = -\bar{\psi}^{ij}{}_{,0} n_\alpha \tag{2.22}$$

Using equations (2.18) to (2.22) in conjunction with equation (2.9) the interaction power between the pth and the qth radiators can be written as

$$\dot{E}_{pq} = \frac{G}{72\pi c^5}\int \left(\dddot{\overset{(p)}{D}}{}^{\alpha\beta} \dddot{\overset{(q)}{D}}{}^{\gamma\delta} n_\alpha n_\beta n_\gamma n_\delta - 4\dddot{\overset{(p)}{D}}{}^{\alpha\beta} \dddot{\overset{(q)}{D}}{}^{\alpha\gamma} n_\beta n_\gamma + 2\dddot{\overset{(p)}{D}}{}^{\alpha\beta} \dddot{\overset{(q)}{D}}{}^{\alpha\beta} \right) d\Omega \tag{2.23}$$

where

$$d\Omega = r^{-2}\, dS \tag{2.24}$$

and

$$\begin{aligned} \overset{(i)}{D}{}^{\alpha\beta} &= \overset{(i)}{D}{}^{\alpha\beta}(t - r/c + \overset{(i)}{\underline{L}}.\underline{n}/c) \qquad i = p, q \\ &= \frac{1}{c^2}(3\overset{(i)}{d}{}^{\alpha\beta} - \delta_\alpha{}^\beta \overset{(i)}{d}{}^{\gamma\gamma}) \end{aligned} \tag{2.25}$$

are the quadrupole moments of the radiators.

Taking the ith radiator to be periodic† with frequency $\overset{(i)}{\omega}$, then

$$\overset{(i)}{D}{}^{\alpha\beta} = \mathrm{Re}\left\{ \overset{(i)}{A}{}^{\alpha\beta} \exp\left(i\left[\overset{(i)}{\omega}(t - r/c + \overset{(i)}{\underline{L}}.\underline{n}/c) + \overset{(i)}{\gamma} \right]\right)\right\} \tag{2.26}$$

† See Discussion, Section 3.

where $\overset{(i)}{A}{}^{\alpha\beta}$ is a complex amplitude and $\overset{(i)}{\gamma}$ is a phase angle. Thus

$$\overset{(i)}{D}{}^{\alpha\beta} = \operatorname{Re} \overset{(i)}{D}{}_c^{\alpha\beta}(t - r/c + \overset{(i)}{\underline{L}} . \underline{n}/c) \tag{2.27}$$

and

$$\overset{(i)}{D}{}_c^{\alpha\beta}(t - r/c + \overset{(i)}{\underline{L}} . \underline{n}/c) = \{\cos \overset{(i)}{\underline{k}} . \underline{n} + i \sin \overset{(i)}{\underline{k}} . \underline{n}\} \overset{(i)}{D}{}_c^{\alpha\beta}(t - r/c) \tag{2.28}$$

where

$$\overset{(i)}{\underline{k}} = \frac{\overset{(i)}{\omega} \overset{(i)}{\underline{L}}}{c} \tag{2.29}$$

When equations (2.26) to (2.29) are substituted into equation (2.23) for the pth and the qth radiators we obtain terms involving quadratic products of cosines, quadratic products of sines and products of a cosine and a sine. Only those terms involving a product of an even number of components of the unit normal give a non-vanishing contribution (Booth, 1970) and consequently we neglect the odd functions obtained from the products of a cosine and a sine.

A typical quadrupole product in the integrand of equation (2.23) then becomes

$$\begin{aligned} \overset{(p)}{\dddot{D}}{}^{\alpha\beta} \overset{(q)}{\dddot{D}}{}^{\gamma\delta} = {} & \tfrac{1}{2}\left\{\overset{(p)}{\dddot{D}}{}^{\alpha\beta} \overset{(q)}{\dddot{D}}{}^{\gamma\delta} - \overset{(p)}{\bar{\dddot{D}}}{}^{\alpha\beta} \overset{(q)}{\bar{\dddot{D}}}{}^{\gamma\delta}\right\}\left\{\cos\left[\left(\overset{(p)}{\underline{k}} + \overset{(q)}{\underline{k}}\right) . \underline{n}\right]\right\} \\ & + \tfrac{1}{2}\left\{\overset{(p)}{\dddot{D}}{}^{\alpha\beta} \overset{(q)}{\dddot{D}}{}^{\gamma\delta} + \overset{(p)}{\bar{\dddot{D}}}{}^{\alpha\beta} \overset{(q)}{\bar{\dddot{D}}}{}^{\gamma\delta}\right\}\left\{\cos\left[\left(\overset{(p)}{\underline{k}} - \overset{(q)}{\underline{k}}\right) . \underline{n}\right]\right\} \end{aligned} \tag{2.30}$$

where, on the right-hand side of equation (2.30)

$$\begin{aligned} \overset{(i)}{D}{}^{\alpha\beta} &= \overset{(i)}{D}{}^{\alpha\beta}(t - r/c) \qquad i = p, q \\ \overset{(i)}{\bar{D}}{}^{\alpha\beta} &= \overset{(i)}{D}{}^{\alpha\beta}\left(t - r/c + \pi/2\overset{(i)}{\omega}\right) \end{aligned} \tag{2.31}$$

To facilitate computation of the integrals involved in equation (2.23)

$$\begin{aligned} \underline{\rho}_{pq} &= \overset{(p)}{\underline{k}} + \overset{(q)}{\underline{k}} = \rho_{pq} \underline{\hat{n}} \\ {}^*\underline{\rho}_{pq} &= \overset{(p)}{\underline{k}} - \overset{(q)}{\underline{k}} = {}^*\rho_{pq} {}^*\underline{\hat{n}} \end{aligned} \tag{2.32}$$

are defined, where $\underline{\hat{n}}$ and ${}^*\underline{\hat{n}}$ are fixed unit vectors which specify the orientation of O_p and O_q relative to O.†

From equations (2.23) and (2.30) it is seen that

$$\begin{aligned} \underline{\rho}_{pq} \cdot \underline{n} &= \rho_{pq}^{\alpha} n_{\alpha} \; (\equiv \rho_{pq} \hat{n}^{\alpha} n_{\alpha}) \\ {}^*\underline{\rho}_{pq} \cdot \underline{n} &= {}^*\rho_{pq}^{\alpha} n_{\alpha} \; (\equiv {}^*\rho_{pq} {}^*\hat{n}^{\alpha} n_{\alpha}) \end{aligned} \tag{2.33}$$

† O_p and O_q are the centres of mass of radiators p and q respectively, O is the common origin.

In order to evaluate the integral of equation (2.23) it is now required to evaluate integrals of the type

$$\int_{4\pi} \cos(\rho_\omega n_\omega)\, d\Omega$$

$$\int_{4\pi} n_\alpha n_\beta \cos(\rho_\omega n_\omega)\, d\Omega \tag{2.34}$$

and

$$\int_{4\pi} n_\alpha n_\beta n_\gamma n_\delta \cos(\rho_\omega n_\omega)\, d\Omega$$

The results of the integrations are given in the Appendix. Equations (A.6) to (A.8), (2.30) and (2.33), in conjunction with equation (2.23), yield the interaction power between the pth and the qth radiators as†

$$\begin{aligned}
\dot{E}_{pq} = \frac{G}{36c^5}\Bigg\{&\left(\overset{(p)}{\dddot{D}}{}^{\alpha\beta}\overset{(q)}{\dddot{D}}{}^{\alpha\beta} - \overset{(p)}{\dddot{\bar{D}}}{}^{\alpha\beta}\overset{(q)}{\dddot{\bar{D}}}{}^{\alpha\beta}\right)\left[\left(\frac{2}{\rho_{pq}} - \frac{6}{\rho_{pq}^3} + \frac{6}{\rho_{pq}^5}\right)\sin\rho_{pq}\right.\\
&\left.+\left(\frac{4}{\rho_{pq}^2} - \frac{6}{\rho_{pq}^4}\right)\cos\rho_{pq}\right]\\
&+\left(\overset{(p)}{\dddot{D}}{}^{\alpha\gamma}\overset{(q)}{\dddot{D}}{}^{\beta\gamma} - \overset{(p)}{\dddot{\bar{D}}}{}^{\alpha\gamma}\overset{(q)}{\dddot{\bar{D}}}{}^{\beta\gamma}\right)\left[\left(-\frac{4}{\rho_{pq}} + \frac{36}{\rho_{pq}^3} - \frac{60}{\rho_{pq}^5}\right)\sin\rho_{pq}\right.\\
&\left.+\left(-\frac{16}{\rho_{pq}^2} + \frac{60}{\rho_{pq}^4}\right)\cos\rho_{pq}\right]\hat{n}_\alpha\hat{n}_\beta\\
&+\left(\overset{(p)}{\dddot{D}}{}^{\alpha\beta}\overset{(q)}{\dddot{D}}{}^{\gamma\delta} - \overset{(p)}{\dddot{\bar{D}}}{}^{\alpha\beta}\overset{(q)}{\dddot{\bar{D}}}{}^{\gamma\delta}\right)\left[\left(\frac{1}{\rho_{pq}} - \frac{45}{\rho_{pq}^3} + \frac{105}{\rho_{pq}^5}\right)\sin\rho_{pq}\right.\\
&\left.+\left(\frac{10}{\rho_{pq}^2} - \frac{105}{\rho_{pq}^4}\right)\cos\rho_{pq}\right]\hat{n}_\alpha\hat{n}_\beta\hat{n}_\gamma\hat{n}_\delta\\
&+\left(\overset{(p)}{\dddot{D}}{}^{\alpha\beta}\overset{(q)}{\dddot{D}}{}^{\alpha\beta} + \overset{(p)}{\dddot{\bar{D}}}{}^{\alpha\beta}\overset{(q)}{\dddot{\bar{D}}}{}^{\alpha\beta}\right)\left[\left(\frac{2}{{}^*\rho_{pq}} - \frac{6}{{}^*\rho_{pq}^3} + \frac{6}{{}^*\rho_{pq}^5}\right)\sin{}^*\rho_{pq}\right.\\
&\left.+\left(\frac{4}{{}^*\rho_{pq}^2} - \frac{6}{{}^*\rho_{pq}^4}\right)\cos{}^*\rho_{pq}\right]\\
&+\left(\overset{(p)}{\dddot{D}}{}^{\alpha\gamma}\overset{(q)}{\dddot{D}}{}^{\beta\gamma} + \overset{(p)}{\dddot{\bar{D}}}{}^{\alpha\gamma}\overset{(q)}{\dddot{\bar{D}}}{}^{\beta\gamma}\right)\left[\left(-\frac{4}{{}^*\rho_{pq}} + \frac{36}{{}^*\rho_{pq}^3} - \frac{60}{{}^*\rho_{pq}^5}\right)\sin{}^*\rho_{pq}\right.\\
&\left.+\left(-\frac{16}{{}^*\rho_{pq}^2} + \frac{60}{{}^*\rho_{pq}^4}\right)\cos{}^*\rho_{pq}\right]{}^*\hat{n}_\alpha{}^*\hat{n}_\beta\\
&+\left(\overset{(p)}{\dddot{D}}{}^{\alpha\beta}\overset{(q)}{\dddot{D}}{}^{\gamma\delta} + \overset{(p)}{\dddot{\bar{D}}}{}^{\alpha\beta}\overset{(q)}{\dddot{\bar{D}}}{}^{\gamma\delta}\right)\left[\left(\frac{1}{{}^*\rho_{pq}} - \frac{45}{{}^*\rho_{pq}^3} + \frac{105}{{}^*\rho_{pq}^5}\right)\sin{}^*\rho_{pq}\right.\\
&\left.+\left(\frac{10}{{}^*\rho_{pq}^2} - \frac{105}{{}^*\rho_{pq}^4}\right)\cos{}^*\rho_{pq}\right]{}^*\hat{n}_\alpha{}^*\hat{n}_\beta{}^*\hat{n}_\gamma{}^*\hat{n}_\delta\Bigg\}
\end{aligned} \tag{2.35}$$

† $\dot{E}_{pq}$ could be evaluated to higher multipole order by retaining higher order terms in equation (2.18).

The total interaction power of a system of N such radiators is then

$$ {}_{\mathrm{int}}\dot{E} = \tfrac{1}{2} \sum_{\substack{p,q=1 \\ p \neq q}}^{N} \dot{E}_{pq} \tag{2.36}$$

If in equation (2.35) the limit is taken as $\rho_{pq} \to 0$ and $^*\rho_{pq} \to 0$ then

$$ \lim_{\substack{^*\rho_{pq} \\ \rho_{pq}}\} \to 0} \dot{E}_{pq} = \frac{-2G}{45c^5} \overset{(p)}{\dddot{D}}{}^{\alpha\beta} \overset{(q)}{\dddot{D}}{}^{\alpha\beta} \tag{2.37}$$

Letting $p = q$ in equation (2.37) yields twice the energy-loss rate from the pth radiator in the absence of interaction. Consequently the total energy-loss rate for this galactic model is

$$ {}_{\mathrm{tot}}\dot{E} = \tfrac{1}{2} \sum_{p,q=1}^{N} \dot{E}_{pq} \tag{2.38}$$

where

$$ \dot{E}_{pp} = \lim_{\substack{^*\rho_{pq} \\ \rho_{pq}}\} \to 0} \dot{E}_{pq} \Big|_{p=q} \tag{2.39}$$

3. *Discussion*

The model proposed is limited to weak fields and consequently takes no account of the existence of black holes. It does seem, however, reasonable to assume that, black holes apart, the galactic core consists of a large number of stars exhibiting 'classical' astrophysical phenomena and for this situation the model gives an adequate description of 'first order' effects as far as gravitational radiation is concerned. For this reason it is hoped that the results obtained in this paper will be useful for the interpretation of gravitational radiation experiments and for the computation of a reasonable lower bound to the gravitational energy flux from the galactic core or from the galaxy as a whole. No attempt has, as yet, been made to produce numbers from the results but as a guide to orders of magnitude it is easily seen that the total energy radiated from a system of similar radiators is approximately proportional to the square of the number of radiators; the constant of proportionality having an order of magnitude comparable to that of the radiation emitted by a single source in the absence of interaction.

The assumption of a system of periodic radiators is not as restrictive as it might at first appear as it is always possible (under appropriate conditions) to Fourier analyse a non-periodic behaviour over a given time interval into a series of periodic harmonics.

It is hoped to complete this work by computing the linear and angular momenta radiated from this model in anticipation of more sophisticated experimental techniques and to give a more complete physical picture of the system.

APPENDIX

The integrals presented here are extensions of the integrals tabulated by Booth (1970).

The integrals in question are

$$\frac{1}{4\pi}\int_{4\pi}\cos(\rho_\omega n_\omega)\,d\Omega \qquad \rho_\omega = \rho\hat{n}_\omega \tag{A.1}$$

$$\frac{1}{4\pi}\int_{4\pi} n_\alpha n_\beta \cos(\rho_\omega n_\omega)\,d\Omega \tag{A.2}$$

$$\frac{1}{4\pi}\int_{4\pi} n_\alpha n_\beta n_\gamma n_\delta \cos(\rho_\omega n_\omega)\,d\Omega \tag{A.3}$$

which are evaluated by means of the identity (Fock, 1964)

$$\frac{1}{4\pi}\int_{4\pi} (a_\omega n_\omega)^{2p}\,d\Omega = \frac{(a_\omega a_\omega)^p}{(2p+1)} \tag{A.4}$$

In each integral we expand $\cos(\rho_\omega n_\omega)$ to be

$$\cos(\rho_\omega n_\omega) = \sum_{m=0}^{\infty}\sum_{r=0}^{2m}\sum_{s=0}^{2m-r} {}^{2m}C_r{}^{2m-r}C_s(\rho_1 n_1)^{2m-r-s}(\rho_2 n_2)^r(\rho_3 n_3)^s \tag{A.5}$$

Substitution of (A.5) into (A.1) to (A.3) and the use of (A.4) yields

$$\frac{1}{4\pi}\int_{4\pi}\cos(\rho_\omega n_\omega) = \frac{\sin\rho}{\rho} \tag{A.6}$$

$$\begin{aligned}\frac{1}{4\pi}\int_{4\pi} n_\alpha n_\beta \cos(\rho_\omega n_\omega)\,d\Omega &= a\,\delta_{\alpha\beta}\\ &+ b\{\hat{n}_1{}^2\delta_{\alpha1}\delta_{\beta1} + \hat{n}_2{}^2\delta_{\alpha2}\delta_{\beta2} + \hat{n}_3{}^2\delta_{\alpha3}\delta_{\beta3}\\ &+ \hat{n}_1\hat{n}_2(\delta_{\alpha1}\delta_{\beta2}+\delta_{\alpha2}\delta_{\beta1}) + \hat{n}_1\hat{n}_3(\delta_{\alpha1}\delta_{\beta3}+\delta_{\alpha3}\delta_{\beta1})\\ &+ \hat{n}_2\hat{n}_3(\delta_{\alpha2}\delta_{\beta3}+\delta_{\alpha3}\delta_{\beta2})\}\end{aligned} \tag{A.7}$$

$$\begin{aligned}\frac{1}{4\pi}\int_{4\pi} n_\alpha n_\beta n_\gamma n_\delta \cos(\rho_\omega n_\omega)\,d\Omega &= -b(\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma})\\ &+ c\{\delta_{\alpha\beta}(\hat{n}_1{}^2\delta_{\gamma1}\delta_{\delta1} + \hat{n}_2{}^2\delta_{\gamma2}\delta_{\delta2} + \hat{n}_3{}^2\delta_{\gamma3}\delta_{\delta3})\\ &+ \delta_{\alpha\gamma}(\hat{n}_1{}^2\delta_{\beta1}\delta_{\delta1} + \hat{n}_2{}^2\delta_{\beta2}\delta_{\delta2} + \hat{n}_3{}^2\delta_{\beta3}\delta_{\delta3})\\ &+ \delta_{\alpha\delta}(\hat{n}_1{}^2\delta_{\beta1}\delta_{\gamma1} + \hat{n}_2{}^2\delta_{\beta2}\delta_{\gamma2} + \hat{n}_3{}^2\delta_{\beta3}\delta_{\gamma3})\\ &+ \delta_{\beta\gamma}(\hat{n}_1{}^2\delta_{\alpha1}\delta_{\delta1} + \hat{n}_2{}^2\delta_{\alpha2}\delta_{\delta2} + \hat{n}_3{}^2\delta_{\alpha3}\delta_{\delta3})\\ &+ \delta_{\beta\delta}(\hat{n}_1{}^2\delta_{\alpha1}\delta_{\gamma1} + \hat{n}_2{}^2\delta_{\alpha2}\delta_{\gamma2} + \hat{n}_3{}^2\delta_{\alpha3}\delta_{\gamma3})\\ &+ \delta_{\gamma\delta}(\hat{n}_1{}^2\delta_{\alpha1}\delta_{\beta1} + \hat{n}_2{}^2\delta_{\alpha2}\delta_{\beta2} + \hat{n}_3{}^2\delta_{\alpha3}\delta_{\beta3})\\ &+ \delta_{\alpha\beta}(\hat{n}_1\hat{n}_2[\delta_{\gamma1}\delta_{\delta2}+\delta_{\gamma2}\delta_{\delta1}] + \hat{n}_1\hat{n}_3[\delta_{\gamma1}\delta_{\delta3}+\delta_{\gamma3}\delta_{\delta1}]\\ &\quad + \hat{n}_2\hat{n}_3[\delta_{\gamma2}\delta_{\delta3}+\delta_{\gamma3}\delta_{\delta2}])\end{aligned}$$

$$
\begin{aligned}
&+ \delta_{\alpha\gamma}(\hat{n}_1 \hat{n}_2[\delta_{\beta 1} \delta_{\delta 2} + \delta_{\beta 2} \delta_{\delta 1}] + \hat{n}_1 \hat{n}_3[\delta_{\beta 1} \delta_{\delta 3} + \delta_{\beta 3} \delta_{\delta 1}] \\
&\quad + \hat{n}_2 \hat{n}_3[\delta_{\beta 2} \delta_{\delta 3} + \delta_{\beta 3} \delta_{\delta 2}]) \\
&+ \delta_{\alpha\delta}(\hat{n}_1 \hat{n}_2[\delta_{\beta 1} \delta_{\gamma 2} + \delta_{\beta 2} \delta_{\gamma 1}] + \hat{n}_1 \hat{n}_3[\delta_{\beta 1} \delta_{\gamma 3} + \delta_{\beta 3} \delta_{\gamma 1}] \\
&\quad + \hat{n}_2 \hat{n}_3[\delta_{\beta 2} \delta_{\gamma 3} + \delta_{\beta 3} \delta_{\gamma 2}]) \\
&+ \delta_{\beta\gamma}(\hat{n}_1 \hat{n}_2[\delta_{\alpha 1} \delta_{\delta 2} + \delta_{\alpha 2} \delta_{\delta 1}] + \hat{n}_1 \hat{n}_3[\delta_{\alpha 1} \delta_{\delta 3} + \delta_{\alpha 3} \delta_{\delta 1}] \\
&\quad + \hat{n}_2 \hat{n}_3[\delta_{\alpha 2} \delta_{\delta 3} + \delta_{\alpha 3} \delta_{\delta 2}]) \\
&+ \delta_{\beta\delta}(\hat{n}_1 \hat{n}_2[\delta_{\alpha 1} \delta_{\gamma 2} + \delta_{\alpha 2} \delta_{\gamma 1}] + \hat{n}_1 \hat{n}_3[\delta_{\alpha 1} \delta_{\gamma 3} + \delta_{\alpha 3} \delta_{\gamma 1}] \\
&\quad + \hat{n}_2 \hat{n}_3[\delta_{\alpha 2} \delta_{\gamma 3} + \delta_{\alpha 3} \delta_{\gamma 2}]) \\
&+ \delta_{\gamma\delta}(\hat{n}_1 \hat{n}_2[\delta_{\alpha 1} \delta_{\beta 2} + \delta_{\alpha 2} \delta_{\beta 1}] + \hat{n}_1 \hat{n}_3[\delta_{\alpha 1} \delta_{\beta 3} + \delta_{\alpha 3} \delta_{\beta 1}] \\
&\quad + \hat{n}_2 \hat{n}_3[\delta_{\alpha 2} \delta_{\beta 3} + \delta_{\alpha 3} \delta_{\beta 2}])\} \\
&+ d\{(\hat{n}_1{}^3 \delta_{\alpha 1} \delta_{\beta 1} \delta_{\gamma 1} + \hat{n}_2{}^3 \delta_{\alpha 2} \delta_{\beta 2} \delta_{\gamma 2} + \hat{n}_3{}^3 \delta_{\alpha 3} \delta_{\beta 3} \delta_{\gamma 3}) \\
&\quad \times (\hat{n}_1 \delta_{\delta 1} + \hat{n}_2 \delta_{\delta 2} + \hat{n}_3 \delta_{\delta 3}) \\
&+ [\hat{n}_1{}^2 \hat{n}_2(\delta_{\alpha 1}[\delta_{\beta 1} \delta_{\gamma 2} + \delta_{\beta 2} \delta_{\gamma 1}] + \delta_{\alpha 2} \delta_{\beta 1} \delta_{\gamma 1}) \\
&+ \hat{n}_2{}^2 \hat{n}_1(\delta_{\alpha 2}[\delta_{\beta 1} \delta_{\gamma 2} + \delta_{\beta 2} \delta_{\gamma 1}] + \delta_{\alpha 1} \delta_{\beta 2} \delta_{\gamma 2}) \\
&+ \hat{n}_1{}^2 \hat{n}_3(\delta_{\alpha 1}[\delta_{\beta 1} \delta_{\gamma 3} + \delta_{\beta 3} \delta_{\gamma 1}] + \delta_{\alpha 3} \delta_{\beta 1} \delta_{\gamma 1}) \\
&+ \hat{n}_3{}^2 \hat{n}_1(\delta_{\alpha 3}[\delta_{\beta 1} \delta_{\gamma 3} + \delta_{\beta 3} \delta_{\gamma 3}] + \delta_{\alpha 1} \delta_{\beta 3} \delta_{\gamma 3}) \\
&+ \hat{n}_2{}^2 \hat{n}_3(\delta_{\alpha 2}[\delta_{\beta 2} \delta_{\gamma 3} + \delta_{\beta 3} \delta_{\gamma 2}] + \delta_{\alpha 3} \delta_{\beta 2} \delta_{\gamma 2}) \\
&+ \hat{n}_3{}^2 \hat{n}_2(\delta_{\alpha 3}[\delta_{\beta 2} \delta_{\gamma 3} + \delta_{\beta 3} \delta_{\gamma 2}] + \delta_{\alpha 2} \delta_{\beta 3} \delta_{\gamma 3}) \\
&+ \hat{n}_1 \hat{n}_2 \hat{n}_3(\delta_{\alpha 1}[\delta_{\beta 2} \delta_{\gamma 3} + \delta_{\beta 3} \delta_{\gamma 2}] + \delta_{\alpha 2}[\delta_{\beta 3} \delta_{\gamma 1} + \delta_{\beta 1} \delta_{\gamma 3}] \\
&+ \delta_{\alpha 3}[\delta_{\beta 1} \delta_{\gamma 2} + \delta_{\beta 2} \delta_{\gamma 1}])] (\hat{n}_1 \delta_{\delta 1} + \hat{n}_2 \delta_{\delta 2} + \hat{n}_3 \delta_{\delta 3})\} \qquad \text{(A.8)}
\end{aligned}
$$

where

$$
\begin{aligned}
a &= (-\rho^{-2} \cos\rho + \rho^{-3} \sin\rho) \\
b &= (\rho^{-1} \sin\rho + 3\rho^{-2} \cos\rho - 3\rho^{-3} \sin\rho) \qquad \text{(A.9)} \\
c &= (-\rho^{-2} \cos\rho + 6\rho^{-3} \sin\rho + 15\rho^{-4} \cos\rho - 15\rho^{-5} \sin\rho) \\
d &= (\rho^{-1} \sin\rho + 10\rho^{-2} \cos\rho - 45\rho^{-3} \sin\rho - 105\rho^{-4} \cos\rho + 105\rho^{-5} \sin\rho)
\end{aligned}
$$

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